

# Quantifying geometric measure of entanglement by mean value of spin and spin correlations for pure and mixed states

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## Abstract

We quantify the geometric measure of entanglement in terms of mean values of observables of entangled system. For pure states we find the relation of geometric measure of entanglement with the mean value of spin one-half for the system composed of spin and arbitrary quantum system. The geometric measure of entanglement for mixed states of rank-2 is studied as well. We find the explicit expression for geometric entanglement and the relation of entanglement in this case with the values of spin correlations. These results allow to find experimentally the value of entanglement by measuring a value of the mean spin and the spin correlations for pure and mixed states, respectively. The obtained results are applied for calculation of entanglement during the evolution in spin chain with Ising interaction, two-spin Ising

model in transverse fluctuating magnetic field, Schrödinger cat in fluctuating magnetic field.

**Keywords:** *entanglement, geometric measure of entanglement, spin correlations, Ising model, Schrödinger cat*

## 1 Introduction

Quantification of entanglement is one of the principal challenges in quantum information theory [1, 2]. Among the natural entanglement measures there is the geometric measure of entanglement proposed by Shimony [3]. Its properties for multiqubit systems were studied by Brody and Hughston [4] and also Wei and Goldbart [5]. A comparison of different definitions of the geometric measure of entanglement can be found in [6].

The geometric measure of entanglement is defined as a minimal squared distance between an entangled state  $|\psi\rangle$  and a set of separable states  $|\psi_s\rangle$

$$E(|\psi\rangle) = \min_{|\psi_s\rangle} (1 - |\langle\psi|\psi_s\rangle|^2) = 1 - \max_{|\psi_s\rangle} |\langle\psi|\psi_s\rangle|^2, \quad (1)$$

where  $1 - |\langle\psi|\psi_s\rangle|^2$  is the squared distance of Fubini-Study. Note that despite its simple definition it involves a nontrivial minimization procedure over separable states.

In the case of mixed states the entanglement can be defined in terms of the convex roof construction

$$E(\rho) = \min \sum_i p_i E(|\psi_i\rangle), \quad (2)$$

where minimization is done over all possible decompositions of density matrix with respect to pure quantum states

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \quad \sum_i p_i = 1. \quad (3)$$

The essential question is what is the way to measure the entanglement directly. Many methods and schemes were proposed for this purpose [7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. In present paper we study geometric measure of entanglement for pure and mixed states and find the relation of entanglement with mean value observables. Namely, for pure states of a spin with arbitrary quantum system we obtain exactly the relation of the entanglement

with mean value of spin. In the case of mixed states of rank-2 we find the explicit expression for geometric entanglement and the relation of the entanglement with spin correlations for special cases of rank-2 mixed states. These mean values are experimentally measurable. Therefore, our results give an additional possibility for direct experimental measurement of degree of entanglement. From the other hand the obtained results give also possibility to find in explicit form geometric entanglement for different quantum systems.

Note also that for mixed quantum states, even bipartite mixed states of rank-2, many questions remain opened (see for instance [17] and references therein). Therefore study of entanglement in mixed states of rank-2 remains interesting and actual.

This paper is organized as follows: In Section 2 we find the relation of geometric entanglement of spin with arbitrary quantum system in pure state with mean value of spin. In Section 3 we study the geometric measure of entanglement for rank-2 mixed states and find relation of entanglement with spin correlations. In section 4 we apply the results obtained in section 2 and 3 for calculation of geometric entanglement during the pure evolution in spin chain and for calculation of geometric entanglement during the evolution of ensemble of two-spin systems in fluctuating magnetic field and for calculation of entanglement during the evolution and the decoherence of Schrödinger cat. And finally, the conclusions are presented in Section 5. The minimization procedure over separable states for geometric entanglement in mixed state is presented in Appendix.

## 2 Characterizing entanglement of spin with arbitrary quantum system by mean value of spin

In general, the pure quantum state of spin one-half (or qubit) which can be entangled with other arbitrary quantum system in pure state reads

$$|\psi\rangle = a|\uparrow\rangle|\phi_1\rangle + b|\downarrow\rangle|\phi_2\rangle, \quad (4)$$

here  $|\phi_1\rangle$  and  $|\phi_2\rangle$  are arbitrary state vectors of quantum system entangled with a spin, constants  $a, b$  are real and positive, phase multipliers can be included into  $|\phi_1\rangle$  and  $|\phi_2\rangle$ , which satisfy normalization conditions  $\langle\phi_1|\phi_1\rangle =$

$\langle\phi_2|\phi_2\rangle = 1$ . Note that in general this functions are not orthogonal  $\langle\phi_1|\phi_2\rangle \neq 0$ .

Arbitrary state vector of spin, interacting with some quantum system, can be represented by the Schmidt decomposition

$$|\psi\rangle = \lambda_1|\alpha_1\rangle|\tilde{\phi}_1\rangle + \lambda_2|\alpha_2\rangle|\tilde{\phi}_2\rangle, \quad (5)$$

where  $|\alpha_1\rangle, |\alpha_2\rangle$  are two orthogonal states of spin

$$|\alpha_1\rangle = \frac{|\uparrow\rangle + \alpha|\downarrow\rangle}{\sqrt{1+|\alpha|^2}}, \quad |\alpha_2\rangle = \frac{\alpha^*|\uparrow\rangle - |\downarrow\rangle}{\sqrt{1+|\alpha|^2}}, \quad (6)$$

and  $|\tilde{\phi}_1\rangle, |\tilde{\phi}_2\rangle$  are two orthogonal states of arbitrary quantum system interacting with the spin,  $\langle\tilde{\phi}_1|\tilde{\phi}_2\rangle = 0$ . Constants  $\lambda_1, \lambda_2$  are real and positive satisfying normalization condition  $\lambda_1^2 + \lambda_2^2 = 1$ .

The geometric measure of entanglement is related with maximum value of squared Schmidt coefficients  $(\lambda_1, \lambda_2)$  [18] namely,

$$E(|\psi\rangle) = 1 - \max(\lambda_1^2, \lambda_2^2). \quad (7)$$

It turns out that Schmidt coefficients  $\lambda_1, \lambda_2$  are related with the mean value of spin. To show this, let us calculate squared mean value of spin

$$\langle\sigma\rangle^2 = (\lambda_1^2 - \lambda_2^2)^2 = (1 - 2\lambda_1^2)^2 = (1 - 2\lambda_2^2)^2. \quad (8)$$

From this relation we have

$$\lambda_{1,2}^2 = \frac{1}{2}(1 \pm |\langle\sigma\rangle|), \quad (9)$$

where  $|\langle\sigma\rangle| = \sqrt{\langle\sigma\rangle^2}$ . Hence, the geometric measure of entanglement given by (7) reads

$$E(|\psi\rangle) = \frac{1}{2}(1 - |\langle\sigma\rangle|). \quad (10)$$

The entanglement of spin with other quantum system is entirely determined by the mean value of spin. Note that we do not need an explicit expression for the Schmidt decomposition in order to calculate the entanglement using (10). It is only important that this decomposition exists.

When a spin state  $|\chi\rangle$  is separable from a state of other system  $|\phi\rangle$

$$|\psi\rangle = |\chi\rangle|\phi\rangle, \quad |\chi\rangle = a|\uparrow\rangle + b|\downarrow\rangle \quad (11)$$

then

$$\langle\sigma\rangle^2 = \langle\chi|\sigma|\chi\rangle^2 = 1. \quad (12)$$

Thus, in this case  $E = 0$ , as expected. Note also, that maximal entanglement of spin system with other quantum system is achieved for configuration with vanishing mean value of spin,  $\langle\sigma\rangle^2 = 0$ . As follows from (10) spin and quantum system are separable when  $|\langle\sigma\rangle| = 1$ .

So, we can establish the value of entanglement of spin with other quantum system by measuring local properties of quantum system in pure state, namely mean value of spin.

### 3 Entanglement of rank-2 mixed states and its relation to mean value of spin correlations

We consider special cases of mixed states of two spins. The first one is the case of mixed states with density matrix (3) where  $|\psi_i\rangle$  are given on subspace spanned by vectors  $|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle$ . The second one is the case of mixed states with density matrix (3) where  $|\psi_i\rangle$  are given on subspace spanned by vectors  $|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle$ . For these two cases of rank-2 mixed states we are able to express entanglement over the mean value of spin correlations.

Let us consider in details the first case for which arbitrary vector of pure state can be written in the form similar to spin-1/2 state vector

$$|\psi\rangle = \cos\frac{\theta}{2}|\uparrow\uparrow\rangle + \sin\frac{\theta}{2}e^{i\phi}|\downarrow\downarrow\rangle, \quad (13)$$

where we introduce the notation

$$|\uparrow\uparrow\rangle = |\uparrow\downarrow\rangle, \quad |\downarrow\downarrow\rangle = |\downarrow\uparrow\rangle. \quad (14)$$

Moreover we can introduce the analog of Pauli operators acting on this subspace

$$\Sigma^x = \sigma_1^x \sigma_2^x, \quad \Sigma^y = \sigma_1^y \sigma_2^x, \quad \Sigma^z = \sigma_1^z \sigma_2^0, \quad (15)$$

where  $\sigma_i^\alpha$  Pauli operators for spin  $i$ . Note, we use traditional notation i.e. when dedicated index enumerates different systems we assume tensor product e.g.  $\sigma_1^x \sigma_2^x = \sigma_1^x \otimes \sigma_2^x$ . One can verify that  $\Sigma^x, \Sigma^y, \Sigma^z$  satisfy all properties of Pauli matrices and act on  $|\uparrow\rangle, |\downarrow\rangle$  in the same way as Pauli matrices act on  $|\uparrow\rangle, |\downarrow\rangle$ . Matrix representation of introduced operators (15) on subspace  $|\uparrow\rangle, |\downarrow\rangle$  reads

$$\Sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \Sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \Sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (16)$$

Density matrix for a pure quantum state (13) can be written in the form

$$\rho = \frac{1}{2}(1 + \mathbf{a} \cdot \mathbf{\Sigma}), \quad (17)$$

where  $\mathbf{a}$  plays the role of Bloch vector with  $|\mathbf{a}| = 1$ , direction of this vector is given by spherical angles  $\theta$  and  $\phi$ .

Density matrix of mixed state for considered subspace reads

$$\rho = \sum_i p_i \rho_i = \frac{1}{2}(1 + \sum_i p_i \mathbf{a}_i \cdot \mathbf{\Sigma}), \quad (18)$$

where  $\rho_i$  is density matrix corresponding to a pure state

$$|\psi_i\rangle = \cos \frac{\theta_i}{2} |\uparrow\rangle + \sin \frac{\theta_i}{2} e^{i\phi_i} |\downarrow\rangle, \quad (19)$$

unit vector  $\mathbf{a}_i$  is defined by angles  $\theta_i$  and  $\phi_i$ . Density matrix of mixed state (18) has the same form as pure state density matrix (17) with

$$\mathbf{a} = \sum_i p_i \mathbf{a}_i, \quad (20)$$

where  $\mathbf{a}$  entirely determines density matrix,  $|\mathbf{a}| \leq 1$  correspond to mixed states and  $|\mathbf{a}| = 1$  corresponds to pure ones.

According to (10) the geometric entanglement of one spin with other in pure state  $|\psi_i\rangle$  reads

$$E(|\psi_i\rangle) = \frac{1}{2}(1 - |\langle \psi_i | \boldsymbol{\sigma}_1 | \psi_i \rangle|). \quad (21)$$

It can be seen that for an arbitrary two-spin quantum state we have  $\langle \boldsymbol{\sigma}_1 \rangle^2 = \langle \boldsymbol{\sigma}_2 \rangle^2$ . So, in (21) we can use mean value of first spin or mean value of the

second one. It means that the measure of entanglement of one spin with an other one is symmetric with respect to the spin subsystems.

One can find that for state (19) mean values of projections of the first spin are

$$\langle \psi_i | \sigma_1^x | \psi_i \rangle = 0, \quad \langle \psi_i | \sigma_1^y | \psi_i \rangle = 0, \quad \langle \psi_i | \sigma_1^z | \psi_i \rangle = \cos \theta_i = a_i^z. \quad (22)$$

Therefore, the geometric measure of entanglement of two spins in pure state (19) is

$$E(|\psi_i\rangle) = \frac{1}{2} (1 - |\langle \psi_i | \sigma_1^z | \psi_i \rangle|) = \frac{1}{2} (1 - |a_i^z|) \quad (23)$$

and geometric entanglement of two spins in mixed state has the following form

$$E(\rho) = \frac{1}{2} \left( 1 - \max \sum p_i |a_i^z| \right). \quad (24)$$

Now we have the problem to find  $\max p_i \sum |a_i^z|$  over all decomposition of density matrix, namely, decomposition of fixed  $\mathbf{a}$  over  $\mathbf{a}_i$  and  $p_i$  according to (20). Note that  $|\mathbf{a}_i| = 1$  and therefore

$$\sum_i p_i |\mathbf{a}_i| = 1. \quad (25)$$

It is convenient to introduce the vector  $\mathbf{l}_i = p_i \mathbf{a}_i$  in terms of which we look for

$$\max \sum_i |l_i^z| \quad (26)$$

under condition

$$\sum_i \mathbf{l}_i = \mathbf{a}, \quad (27)$$

with constraint

$$\sum_i |\mathbf{l}_i| = 1. \quad (28)$$

Interestingly enough that this problem has geometric interpretation which is useful for its solving (for explicit construction cf. the Appendix). As a result we obtain

$$\max_i \sum |l_i^z| = \sqrt{1 - a^2 \sin^2 \theta}, \quad (29)$$

where  $a = |\mathbf{a}|$ ,  $\theta$  is angle between  $\mathbf{a}$  and  $z$  axis. Because  $\max p_i \sum |a_i^z| = \max \sum_i |l_i^z|$ , finally we get

$$E(\rho) = \frac{1}{2} \left( 1 - \sqrt{1 - a^2 \sin^2 \theta} \right) = \frac{1}{2} \left( 1 - \sqrt{1 - a_x^2 - a_y^2} \right). \quad (30)$$

To express this result by the mean value of spin correlations, note that the components of the Bloch vector

$$a^x = \langle \Sigma^x \rangle = \langle \sigma_1^x \sigma_2^x \rangle, \quad (31)$$

$$a^y = \langle \Sigma^y \rangle = \langle \sigma_1^y \sigma_2^x \rangle, \quad (32)$$

$$a^z = \langle \Sigma^z \rangle = \langle \sigma_1^z \rangle, \quad (33)$$

where  $\langle A \rangle = \text{Sp} A \rho$  is mean value for mixed state. Thus we have

$$E(\rho) = \frac{1}{2} \left( 1 - \sqrt{1 - \langle \sigma_1^x \sigma_2^x \rangle^2 - \langle \sigma_1^y \sigma_2^x \rangle^2} \right). \quad (34)$$

Thus, the geometric entanglement of two spins in mixed states formed on subspace spanned by vectors  $|\uparrow\downarrow\rangle, |\downarrow\downarrow\rangle$  can be written in terms of the mean values of spin correlations.

One can verify that result given by (34) is valid also in the case of mixed states on subspace spanned by vectors

$$|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle, \quad |\downarrow\downarrow\rangle = |\downarrow\downarrow\rangle. \quad (35)$$

Analyzing (30) we can conclude that the maximally entangled states with  $E = 1/2$  can be obtained only in the case of  $|\mathbf{a}| = 1$ , which corresponds to pure states. An arbitrary mixed state has magnitude of entanglement less than the maximal value  $1/2$ . From (34) follows that two-spin states in considered family of mixed states are non entangled when  $\langle \sigma_1^x \sigma_2^x \rangle = 0$  and  $\langle \sigma_1^y \sigma_2^x \rangle = 0$ .



The obtained result can be generalized for rank-2 mixed state of arbitrary number of spins  $N$ . Let us consider the rank-2 mixed state of  $N$  spins with density matrix (3) where  $|\psi_i\rangle$  are given on subspace spanned by vectors

$$|\uparrow\rangle = |\uparrow\uparrow \dots \uparrow\rangle, \quad |\downarrow\rangle = |\downarrow\downarrow \dots \downarrow\rangle. \quad (36)$$

Now similarly to (15) we can introduce the following analog of Pauli operators acting on this subspace

$$\Sigma^x = \sigma_1^x \sigma_2^x \dots \sigma_N^x, \quad \Sigma^y = \sigma_1^y \sigma_2^y \dots \sigma_N^y, \quad \Sigma^z = \sigma_1^z \sigma_2^z \dots \sigma_N^z. \quad (37)$$

Therefore geometric entanglement (30) derived for two spins in mixed state is suitable also for the case of  $N$  spins. Difference is only that  $\Sigma$  operators now is given by (37) instead of (15). Note that in the case of  $N$  spin (30) describe the geometric entanglement of first spin with others. When we want to find geometric entanglement, for instance, second spin with others then operators (37) must be changed to

$$\Sigma^x = \sigma_1^x \sigma_2^x \sigma_3^x \dots \sigma_N^x, \quad \Sigma^y = \sigma_1^y \sigma_2^y \sigma_3^y \dots \sigma_N^y, \quad \Sigma^z = \sigma_1^z \sigma_2^z \sigma_3^z \dots \sigma_N^z. \quad (38)$$

In the same way we can find entanglement of arbitrary spin with others.

## 4 Calculation of entanglement in particular cases of spin systems

### 4.1 Entanglement in spin chain. Pure states.

In this section we study the entanglement of one spin with others in a spin chain. Relation (10) between entanglement and mean value of spin turns out to be useful for this purpose. Let us consider the spin chain with Ising Hamiltonian

$$H = J \sum_{i=1}^{N-1} \sigma_i^x \sigma_{i+1}^x, \quad (39)$$

where  $N$  is the number of spins in chain,  $\sigma_i^x$  is the Pauli matrix of  $i$ -th spin. We consider the evolution of spins starting at time  $t = 0$  from a factorized state with zero entanglement

$$|\psi_{t=0}\rangle = |\psi_1\rangle |\psi_2\rangle \dots |\psi_N\rangle, \quad (40)$$

where

$$|\psi_i\rangle = a_i|\uparrow\rangle_i + b_i|\downarrow\rangle_i \quad (41)$$

is the state of  $i$ -th spin.

Interaction with Hamiltonian (39) leads to the appearance of entanglement during the evolution. We study the entanglement of the first spin with other  $N - 1$  spins at time  $t$ . In order to calculate the magnitude of the entanglement we use formula (10) relating entanglement with mean value of spin. In our case it is necessary to calculate the mean value of the first spin, namely

$$\langle \sigma_1 \rangle = \langle \psi(t) | \sigma_1 | \psi(t) \rangle, \quad (42)$$

where vector of state at time  $t$  is given by

$$|\psi(t)\rangle = \exp(-i\omega t \sum_{i=1}^{N-1} \sigma_i^x \sigma_{i+1}^x) |\psi_{t=0}\rangle = \prod_{i=1}^N \exp(-i\omega t \sigma_i^x \sigma_{i+1}^x) |\psi_{t=0}\rangle, \quad (43)$$

here  $\omega = J/\hbar$ . Substituting it into (42) we find that exponents in the operator of evolution which does not contain  $\sigma_1^x$  is canceled. As a result, for the mean value of the first spin we obtain

$$\langle \sigma_1 \rangle = \langle \psi_2 | \langle \psi_1 | e^{i\omega t \sigma_1^x \sigma_2^x} \sigma_1 e^{-i\omega t \sigma_1^x \sigma_2^x} | \psi_1 \rangle | \psi_2 \rangle \quad (44)$$

with components

$$\langle \sigma_1^x \rangle = \langle \sigma_1^x \rangle_0, \quad (45)$$

$$\langle \sigma_1^y \rangle = \cos 2\omega t \langle \sigma_1^y \rangle_0 - \sin 2\omega t \langle \sigma_1^z \rangle_0 \langle \sigma_2^x \rangle_0, \quad (46)$$

$$\langle \sigma_1^z \rangle = \cos 2\omega t \langle \sigma_1^z \rangle_0 + \sin 2\omega t \langle \sigma_1^y \rangle_0 \langle \sigma_2^x \rangle_0, \quad (47)$$

where  $\langle \sigma_i^\alpha \rangle_0 = \langle \psi_i | \sigma_i^\alpha | \psi_i \rangle_0$  is the mean value of  $i$ -th spin ( $i = 1, 2$ ,  $\alpha = x, y, z$ ) in the initial state at  $t = 0$ . Then according to (10) the geometric entanglement of the first spin with others in the spin chain reads

$$E = \frac{1}{2} \left( 1 - \sqrt{\langle \sigma_1^x \rangle_0^2 + (\cos^2 2\omega t + \sin^2 2\omega t \langle \sigma_2^x \rangle_0^2) (\langle \sigma_1^y \rangle_0^2 + \langle \sigma_1^z \rangle_0^2)} \right). \quad (48)$$

It is interesting to note that the entanglement of the first spin with the rest of the spin chain depends only on the mean value of the first and second

spins that is the result of nearest-neighbor interactions in Hamiltonian. One can verify that at  $t = 0$  the entanglement is zero as it must be for factorized state. Really, at  $t = 0$  under the square root we have  $\langle \sigma_1 \rangle_0^2$  that is equal to unity for an arbitrary state of the first spin and therefore  $E = 0$  for the initial state.

Now let us apply (48) for some concrete initial states. Let the state

$$|\psi_i\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_i \pm |\downarrow\rangle_i) \quad (49)$$

is the eigenstate of  $\sigma_i^x$ . In this case the initial state (40) is the eigenstate of Hamiltonian (39). Therefore, the initial state does not change during the evolution and thus entanglement for all times is zero. One can verify that the same result follows from (48). For (49) the mean value of the components for the first spin are  $\langle \sigma_1^x \rangle_0 = \pm 1$ ,  $\langle \sigma_1^y \rangle_0 = \langle \sigma_1^z \rangle_0 = 0$  and according to (48) in this case  $E = 0$ .

Notice that  $E = 0$  when only the second spin is in state (49). Then  $\langle \sigma_2^x \rangle_0^2 = 1$  and under the square root we have  $\langle \sigma_1 \rangle_0^2$  that is equal to unity for an arbitrary state of the first spin and therefore  $E = 0$ . Thus, in order to generate the entanglement between first spin and others the mean value of  $x$ -component of the second spin in the initial state must satisfy condition  $\langle \sigma_2^x \rangle_0^2 \neq 1$ .

Now let us consider the initial state for  $N$  spins

$$|\psi_{t=0}\rangle = |\uparrow\rangle_1 |\uparrow\rangle_2 \cdots |\uparrow\rangle_N. \quad (50)$$

In this case

$$E = \frac{1}{2} (1 - |\cos 2\omega t|). \quad (51)$$

Finally let us stress that the relation between entanglement and mean value of spin (10) plays the crucial role in the calculation of the entanglement during the evolution of spins. As result it is not necessary to find state vector during the evolution explicitly. We can directly calculate the mean value of spin and determine the entanglement.

## 4.2 Entanglement of two spins in fluctuating magnetic field. Mixed states.

In this Section we demonstrate the usefulness of formula (30) for calculation entanglement of two spins in mixed state. For this purpose we consider

ensemble of two-spin systems described by Ising Hamiltonian and placed in transverse fluctuating magnetic field

$$H = B(\sigma_1^z + \sigma_2^z) + J\sigma_1^x\sigma_2^x, \quad (52)$$

here  $B$  is magnetic field. Note that magnetic fields of different magnitudes are applied to different two-spin systems from this ensemble.

Hamiltonian (52) has two invariant subspaces. First one is spanned by vectors  $|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle$  and second one is spanned by vectors  $|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle$ . Eigenvectors of (52) belong to these subspaces.

We consider the following problem. Let at the initial time  $t = 0$  all pairs of spins from the ensemble are in the same separated state

$$|\psi_{t=0}\rangle = |\uparrow\rangle_1 |\uparrow\rangle_2 = |\uparrow\uparrow\rangle. \quad (53)$$

Vector of evolution in this case belongs to the subspace spanned by vectors  $|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle$ . As result of evolution of different two-spin systems in different magnetic fields we obtain mixed state. Our goal is to find entanglement of two spins in this mixed state.

One can also easy verify that for this subspace we have

$$H^2 = J^2 + 4B^2 = \hbar^2(\omega^2 + \Omega^2), \quad (54)$$

here for the convenience we introduce the notations

$$B = \frac{\hbar\Omega}{2}, J = \hbar\omega. \quad (55)$$

As result of (54) evolution operator can be written in the following form

$$e^{-iHt/\hbar} = \cos \Omega_0 t - i \frac{H}{\hbar\Omega_0} \sin \Omega_0 t, \quad (56)$$

where  $\Omega_0 = \sqrt{\omega^2 + \Omega^2}$ . Then, the evolution of two-spin system from the ensemble reads

$$|\psi(t)\rangle = e^{-iHt/\hbar} |\uparrow\uparrow\rangle = (\cos \Omega_0 t - i \frac{\Omega}{\Omega_0} \sin \Omega_0 t) |\uparrow\uparrow\rangle - i \frac{\omega}{\Omega_0} \sin \Omega_0 t |\downarrow\downarrow\rangle. \quad (57)$$

The evolution of ensemble of two-spin systems is described by density matrix

$$\rho = \int d\Omega P(\Omega) |\psi(t)\rangle \langle \psi(t)|, \quad (58)$$

where  $P(\Omega)$  is distribution function of magnitude of magnetic field,  $\Omega$  is related with magnetic field by (55). Substituting (57) into (58) and using  $|\uparrow\uparrow\rangle$  and  $|\downarrow\downarrow\rangle$  as basis vectors, we find density matrix describing evolution of ensemble of two-spin systems in fluctuating magnetic field

$$\begin{aligned} \rho &= \\ &= \begin{pmatrix} \langle \cos^2 \Omega_0 t \rangle_\Omega + \langle \frac{\Omega^2}{\Omega_0^2} \sin^2 \Omega_0 t \rangle_\Omega & \frac{i}{2} \langle \frac{\omega}{\Omega_0} \sin 2\Omega_0 t \rangle_\Omega + \langle \frac{\Omega\omega}{\Omega_0^2} \sin^2 \Omega_0 t \rangle_\Omega \\ -\frac{i}{2} \langle \frac{\omega}{\Omega_0} \sin 2\Omega_0 t \rangle_\Omega + \langle \frac{\Omega\omega}{\Omega_0^2} \sin^2 \Omega_0 t \rangle_\Omega & \langle \frac{\omega^2}{\Omega_0^2} \sin^2 \Omega_0 t \rangle_\Omega \end{pmatrix}, \end{aligned} \quad (59)$$

where  $\langle f(\Omega) \rangle_\Omega = \int d\Omega P(\Omega) f(\Omega)$ . This density matrix can be written in form (17) where

$$a_x = 2 \langle \frac{\Omega\omega}{\Omega_0^2} \sin^2 \Omega_0 t \rangle_\Omega, \quad (60)$$

$$a_y = -\langle \frac{\omega}{\Omega_0} \sin 2\Omega_0 t \rangle_\Omega, \quad (61)$$

$$a_z = 1 - 2 \langle \frac{\omega^2}{\Omega_0^2} \sin^2 \Omega_0 t \rangle_\Omega. \quad (62)$$

Substituting this result into (30) we find explicit expression for geometric entanglement

$$E = \frac{1}{2} \left( 1 - \sqrt{1 - 4 \langle \frac{\Omega\omega}{\Omega_0^2} \sin^2 \Omega_0 t \rangle_\Omega^2 - \langle \frac{\omega}{\Omega_0} \sin 2\Omega_0 t \rangle_\Omega^2} \right). \quad (63)$$

Let us consider the following distribution function

$$P(\Omega) = \frac{1}{2} (\delta(\Omega - \chi) + \delta(\Omega + \chi)). \quad (64)$$

It means that magnetic field has the same magnitude  $|B| = \hbar\chi/2$ , the direction of magnetic field is change only. Namely, with probability 1/2 magnetic field has positive direction along  $z$ -axis and with the same probability negative direction along  $z$ -axis. As a result, the mean value of magnetic field is zero,  $\chi$  characterize the value of fluctuation of magnetic field. In this case geometric entanglement reads

$$E = \frac{1}{2} \left( 1 - \sqrt{1 - \frac{\omega^2 \sin^2 2\sqrt{\chi^2 + \omega^2} t}{\chi^2 + \omega^2}} \right). \quad (65)$$

In the case when fluctuation of magnetic field is zero,  $\chi = 0$ , the model considered in this Section corresponds to the model considered in the previous Section 4.1 for number of spins  $N = 2$ . One can verify that for  $\chi = 0$  equation (65) reproduce (51) as it must be. Note that increasing of fluctuation of magnetic field leads to the decreasing of entanglement. In the limit  $\chi \rightarrow \infty$  when fluctuations of magnetic field tend to infinity the geometric entanglement tends to zero.

Now let us consider Gaussian distribution function

$$P(\Omega) = \frac{\tau}{\sqrt{\pi}} e^{-\tau^2 \Omega^2}. \quad (66)$$

In this case for large time we find the following asymptotic for geometric entanglement

$$E = \frac{\omega \tau^2}{4t} \sin^2(2\omega t + \frac{\pi}{4}), \quad t \rightarrow \infty, \quad (67)$$

that tends to 0 when time  $t \rightarrow \infty$ . Note that for distribution function (64) the entanglement is periodic function in time but for Gaussian distribution function (66) the entanglement tends to zero when time go to infinity. Thus, the behavior of entanglement in time for mixed states essentially depends on distribution function of fluctuating magnetic fields.

Finally let us analyze evolution of system under consideration, starting from the initial state

$$|\psi_{t=0}\rangle = |\uparrow\rangle_1 |\downarrow\rangle_2 = |\downarrow\uparrow\rangle. \quad (68)$$

In this case the vector of evolution belongs to the second subspace spanned by vectors  $|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle$ . Hamiltonian in this subspace satisfies the following relation

$$H^2 = \omega^2. \quad (69)$$

Operator of evolution in this case has the form (56) where instead of  $\Omega_0$  we have  $\omega$ . Note also that action of operator  $B(\sigma_1^z + \sigma_2^z)$  on  $|\uparrow\downarrow\rangle$  or  $|\downarrow\uparrow\rangle$  is zero. Therefore fluctuating magnetic field has not influence on the evolution which is now described by pure state. Making similar calculation as in the first case for entanglement we obtain the result (63) where  $\Omega = 0$  and  $\Omega_0$  is changed to  $\omega$ . As a result for geometric entanglement we obtain the same formula as in (51).

### 4.3 Decoherence of Schrödinger cat and geometric entanglement

Let us consider  $N$ -spin systems placed in fluctuating magnetic field with hamiltonian

$$H = \sum_i^N \frac{\hbar \Omega_i}{2} \sigma_i^z. \quad (70)$$

We suppose that distribution function for magnetic fields acting on different spins is independent

$$P(\Omega_1, \Omega_2, \dots, \Omega_N) = P(\Omega_1)P(\Omega_2)\dots P(\Omega_N) \quad (71)$$

In the initial time  $t = 0$  the system is in pure Schrödinger cat quantum state

$$|\psi_{t=0}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle). \quad (72)$$

One can easily find the density matrix describing the evolution of Schrödinger cat in fluctuating magnetic field

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & \langle e^{-i\Omega t} \rangle_{\Omega}^N \\ \langle e^{i\Omega t} \rangle_{\Omega}^N & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & e^{-Nt^2/4\tau^2} \\ e^{-Nt^2/4\tau^2} & 1 \end{pmatrix}. \quad (73)$$

where we write the explicit expression for density matrix in the case of Gaussian distribution function (66). The decoherence in time is more quicker for larger  $N$ . From (30) we find explicit expression for geometric entanglement of one spin with others in this case

$$E(\rho) = \frac{1}{2} \left( 1 - \sqrt{1 - e^{-Nt^2/2\tau^2}} \right). \quad (74)$$

Thus decoherence leads to decreasing of the entanglement to zero.

## 5 Conclusions

In this paper we have studied the geometric measure of entanglement of spin- $\frac{1}{2}$  with other quantum system for pure and mixed states. The main result of

the present paper is given by (10) for pure states and (30) or (34) for mixed states.

In the case of pure quantum states we have shown that entanglement is entirely determined by the mean value of spin (10). Thus, measuring of the mean value of spin allows to find experimentally the value of entanglement of spin with other quantum system in pure state. It is worth mentioning that spin is maximally entangled with other quantum system when its mean value is zero,  $|\langle \sigma \rangle| = 0$ , and it is separable when  $|\langle \sigma \rangle| = 1$  that follows from (10).

We have also considered entanglement of two spins in mixed states of rank-2 which are defined on subspaces spanned by vectors  $|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle$  or subspace spanned by vectors  $|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle$ . For these cases we have found explicit expression for geometric entanglement and have shown, that the geometric entanglement can be expressed by the mean values of spin correlations (34). This result allows to find experimentally geometric entanglement by measuring corresponding mean value of correlations of two spins in mixed state. Using our result (30) we have concluded that the maximally entangled states with  $E = 1/2$  can be obtained only in the case of  $|\mathbf{a}| = 1$ , which corresponds to pure states. For arbitrary mixed state  $|\mathbf{a}| < 1$ , thus, the magnitude of entanglement is less than the maximal value  $1/2$ . From (34) we have concluded that in the case of  $\langle \sigma_1^x \sigma_2^x \rangle = 0$  and  $\langle \sigma_1^y \sigma_2^y \rangle = 0$  the two-spin mixed states are non-entangled. These results are also generalized on rank-2 mixed system of arbitrary number of spins.

Our results (10), (34) connect the entanglement with observables. Therefore, the present approach provides the effective way of an experimental determination of geometric measure of entanglement for considered pure and mixed states of rank-2.

The relation of the entanglement with the mean value of spin (10) is very useful for the calculation of entanglement. As an example we consider the entanglement of the first spin with others in spin chain during the evolution with the Ising Hamiltonian. For the calculation of entanglement it is not necessary to find state vector during the evolution explicitly. It is enough to find mean value of the first spin and with the help of (10) to find entanglement. In such a way we find in explicit form the geometric entanglement of the first spin with others (48) in the Ising spin chain during the evolution.

We also show the usefulness of formula (30) for calculation geometric entanglement of two spins in mixed state. For this purpose we consider ensemble of two-spin systems described by Ising Hamiltonian and placed in transverse fluctuating magnetic field. Using (30) we find geometric entangle-



ment for this system in explicit form (63). As another example we consider the Schrödinger cat quantum state of  $N$  particles and find geometric entanglement of one spin with others during the evolution and decoherence of this system.

Finally let us show that our explicit expression for geometric entanglement (30) reproduce results presented in [17]. Let us consider one of them presented on Fig.2 in [17]. Namely, the authors state that the Bloch sphere has only one line of separable states: all the states along the line connecting  $|00\rangle$  and  $|11\rangle$  (in our notation  $|\uparrow\uparrow\rangle$  and  $|\downarrow\downarrow\rangle$ ) are separable. This result immediately follows from our result (30). Really, the geometric entanglement  $E(\rho) = 0$  when  $a_x = a_y = 0$  that corresponds to the line connecting  $|\uparrow\uparrow\rangle$  and  $|\downarrow\downarrow\rangle$  on the Bloch sphere. Advantage of our result is that we have explicit expression (30) for geometric entanglement of rank-2 mixed states and which is suitable for arbitrary number of spins. This give a possibility to calculate the value of geometric entanglement for different quantum systems that was demonstrated in our paper.

## Appendix

The question of finding maximum (26) can be reformulated in geometrical terms. The constraint (28) can be interpreted in such a way, that we have an inextensible cord of unit length. The ends of this cord, according to (27), are placed at the beginning and at the end of the vector  $\mathbf{a}$ . The problem of finding maximum (26) corresponds to the pulling of the cord in such a way that the projection of  $K'$  on the  $Z$ -axis is maximal (see fig. 1). The quantity  $\max \sum_i |l_i^z|$  is equal to  $|OK| + |O'K''|$ . The point  $K'$  lays on ellipse, because  $|OK'| + |K'O'| = 1$  as fixed length of cord. The focuses of the ellipse are pointed at  $O$  and  $O'$ . The tangent to the ellipse at point  $K'$  is perpendicular to the  $Z$ -axis. Let us make the construction as is shown in fig.1. We continue line  $OK'$  to the point  $L'$ , where  $L'$  lays on  $O'L'$ , which is parallel to the  $Z$ -axis. Let us show that right triangles  $\triangle O'K''K'$  and  $\triangle L'K'K''$  are equal. It is known that, if a rays' source is placed at one focus of an elliptic mirror, all rays on the plane of the ellipse are reflected to the second focus. This means that the angles  $\widehat{O'K'K''}$  and  $\widehat{OK'K'}$  are equal. According to the construction the angles  $\widehat{OK'K}$  and  $\widehat{L'K'K''}$  are equal as vertical. Therefore, the angle  $\widehat{L'K'K''}$  equals to  $\widehat{O'K'K''}$ . Thus, right triangles  $\triangle O'K''K'$  and  $\triangle L'K'K''$  have equal angles adjacent to the common

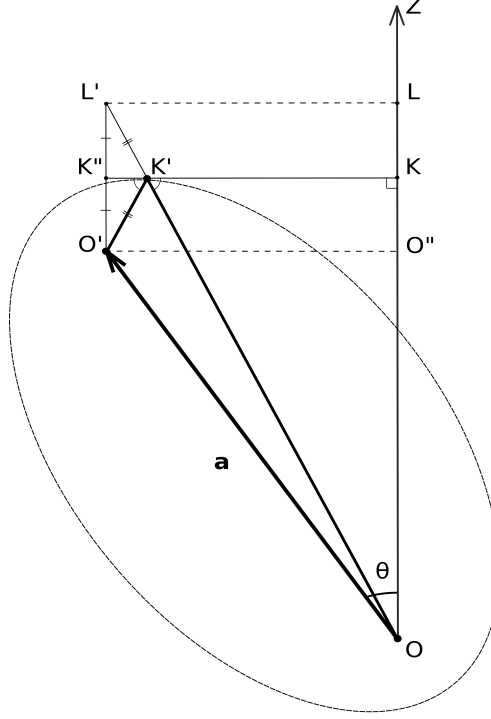


Figure 1: Geometric solution of problem (26).

side. As the consequence, the said triangles are equal. So, the sought quantity reads  $\max \sum_i |l_i^z| = |OK| + |K'L'| = |OL|$ . We can find  $|OL|$  from the right triangle  $\triangle(OL'L)$  as  $|OL| = \sqrt{|OL'|^2 - |LL'|^2}$ . The hypotenuse of triangle  $\triangle(OL'L)$  equals  $|OL'| = |OK'| + |K'L'| = |OK'| + |K'O'| = 1$  as the length of the cord. The leg of this triangle is the following  $|LL'| = |O'O''| = a \sin \theta$ , where  $a = |\mathbf{a}|$ ,  $\theta$  is the angle between  $\mathbf{a}$  and the  $Z$ -axis. Finally, we have

$$|OL| = \sqrt{1 - a^2 \sin^2 \theta}. \quad (75)$$

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